M-IDEAL PRESERVING MAPS AND BANACH-STONE TYPE THEOREMS

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Abstract. We investigate the properties enjoyed by a surjective linear isomorphism between Banach spaces which preserves M-ideals. We say such maps have Property $M$. Under property $M$ we show that if $T$ is a linear isomorphism between affine function spaces $A(K)$ and $A(S)$ and every extreme point of $K$ and $S$ are split faces, then $\partial K$ is facially homeomorphic to $\partial S$. We give examples to show that such a linear isomorphism need not be an isometry and may have arbitrary bound. A key to our result lies in the fact that an M-ideal in $A(K)$ is the annihilator of a closed split face of $K$. Finally, we begin looking at how to characterise the class of isomorphisms which have property $M$.

In this paper we begin by considering the notion of an M-ideal within various Banach space and Banach algebra settings. The principle reason for this is to be able to answer the natural question: what type of mapping preserves an M-ideal? (Within a Banach space setting, by a ‘map’ we always means a surjective linear isomorphism.) Within a ring or algebra structure, the answer is a homomorphism, but if we add a linear structure, what can we say?

We begin with some simple examples showing that such maps need not be isometries and then go on to relate this to a classical problem; namely, we relate M-ideals to a Banach-Stone type theorem for $A(K)$. We conclude this paper by considering candidates for the space of surjective linear isomorphisms satisfying Property $M$. I warmly thank my supervisor, Professor Cho-Ho Chu, QMW, London, for his enlightened discussions and suggestions for sections 2 and 3.

1. Ideals and M-ideals in Banach spaces and Banach Algebras

We begin with a brief survey. Let $E$ be a complex Banach space, then we call $E$ a Banach Algebra if for all $x, y \in E$, we have $x \cdot y \in E$ and $E$ is a normed algebra whose norm satisfies the inequality $\|x \cdot y\| \leq \|x\|\|y\|$. If $E \ni e$ where $e$ is the multiplicative identity, $E$ is called unital. A subset $J \subset E$ is called an ideal if $J$ is a subspace and $x, y \in J$ for all $x \in E$ and $y \in J$. Further, $J$ is called maximal if $J$ is a proper ideal ($J \subsetneq E$) and $J$ is not contained in any larger proper ideal. In a commutative Banach algebra, every proper ideal is contained in a maximal ideal and every maximal ideal is closed. Let $X$ be a compact Hausdorff space and let $C_{c}(X)$ denote the usual Banach space of continuous complex-valued functions on $X$ together with the supremum norm.

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A subspace $M$ of $C_C(X)$ a Function space if $M$ is uniformly closed in $C_C(X)$, contains the constant functions and separates the points of $X$. Let $\mathcal{A}$ be a subspace of $C_C(X)$, then we call $\mathcal{A}$ a Function Algebra (also called Uniform algebras) if $\mathcal{A}$ is a function space and a subalgebra of $C_C(X)$.

The closed ideals of $C_C(X)$ are precisely the closed algebra ideals and are characterised as

$$\{ f : f \in C_C(X) \text{ and } f|_E = 0 \}$$

where $E \subset X$ is closed. That is, a closed ideal is the annihilator of a closed subset of $X$.

The maximal ideals in $C_C(X)$ are sets of the form

$$M_p = \{ f \in C_C(X) : f(p) = 0 \}$$

for any $p \in X$, and it is well-known that $p \mapsto M_p$ gives a 1-1 correspondence between $X$ and the maximal ideal space $\{ M \in C_C(X) : M \text{ is a maximal ideal} \}$ with the Gelfand topology.

An Ideal is an algebraic object and a comparable linear structure is an $M$-ideal.

Let $E$ and $F$ be a real Banach spaces.

**Definition 1.1.** Let $P : E^* \to E^*$ be a continuous projection, that is, a continuous linear map satisfying $P^2 = P$. We call $P$ an L-projection if

$$\|x\| = \|Px\| + \|x - Px\| \quad \forall x \in E^*.$$  

If $P$ is an L-projection, then $Id - P$ is also an L-projection where $Id$ is the identity operator. Moreover $P$ is a contraction, that is, $\|Px\| \leq \|x\|$. 

**Definition 1.2.** A closed subspace $J$ of $E$ is called an M-ideal if the annihilator

$$J^\perp = \{ f \in E^* : f(J) = 0 \}$$

of $J$ is the range of an L-projection on $E^*$, namely,

$$J^\perp = P(E^*)$$

for an L-projection $P$ on $E^*$.

If $P$ is an L-projection on $A(K)^*$ and $x$ is a state of $A(K)$, namely, $x(1) = 1 = \|x\|$, then since $|(Px)(1)| \leq \|Px\| \leq \|x\|$ and $\|(Id - P)x(1)\| \leq \|x - Px\|$, we have

$$1 = x(1) = (Px)(1) + (Id - P)x(1) = \|x\|$$

which gives

$$\|Px\| + \|x - Px\| \geq \|(Px)(1) + (Id - P)x(1)\| \geq 1$$

which gives

$$\|Px\| = \|(Px)(1)\| = \|Px\|.$$  

In $C_C(X)$ M-ideals are exactly the closed algebra ideals; they are the annihilators of closed subsets of $X$.

Alfsen and Effros [3] have characterised M-ideals in a Banach space $E$ in terms of the 3-ball property.
Definition 1.3. A linear subspace $J$ of $E$ satisfies the 3-ball property if given 3 open balls $B_1, B_2, B_3$ in $E$, for which $B_1 \cap B_2 \cap B_3 \neq \emptyset$, and $B_i \cap J \neq \emptyset$, for $i = 1, 2, 3$, then $B_1 \cap B_2 \cap B_3 \cap J \neq \emptyset$.

The following characterisation is due to Alfsen and Effros [3]:

Theorem 1.1. Suppose $J$ is a closed subspace of a Banach space $E$. Then the following are equivalent:

a) $J$ is an M-ideal;

b) $J$ satisfies the 3-ball property.

This implies that an isometry between Banach spaces preserves M-ideals. We note that an M-ideal-preserving linear isomorphism need not be an isometry as the following examples will show. Before we are able to discuss the examples, we briefly recall some background.

Throughout this paper $K$ and $S$ are compact convex sets.

Recall that a convex subset $F$ of $K$ is called a face of $K$ if $\lambda x + (1 - \lambda)y \in F$ for $x$ and $y$ in $K$ and $\lambda \in (0, 1)$, implies that both $x, y \in F$, equivalently, if $K \setminus F$ is convex. If $F$ is a face, a set $F'$ in $K$ is called complementary to $F$ if $F \cap F' = \emptyset$ and $K = \text{co}(F \cup F')$. Thus each $x$ in $K$ has a decomposition relative to $(F, F')$ namely $x = \lambda y + (1 - \lambda)z$ for some $y \in F$, $z \in F'$ and $\lambda$ in $[0, 1]$. If $F'$ is a face we call $(F, F')$ a pair of complementary faces, and if further $\lambda$ is unique, $F$ is called a parallel face. If in addition $y$ and $z$ are unique, then $F$ is called a split face and $(F, F')$ is called a pair of complementary split faces. The facial topology on $\partial K$ is defined by taking

$$\{F \cap \partial K : F \text{ is a closed split face of } K\}$$

as the family of all closed sets. The facial topology is weaker than the relative topology on $\partial K$; it is always $T_0$, but is $T_2$ if and only if $K$ is a Bauer simplex [2]. However $\partial K$ is compact in the facial topology. ([1, page 143].)

The following lemma which is straightforward to prove shows that there is a natural characterisation of an M-ideal in $A(K)$, namely as the annihilator of a closed split face of $K$ (c.f. [6]). (We omit the proof for convenience.)

Lemma 1.2. Let $J$ be a closed subspace of $A(K)$. Then $J$ is an M-ideal if and only if $J = F^\perp$ for $F$ a closed split face of $K$. 
2. Examples of M-ideal-Preserving maps

We now give some examples to investigate the structure of linear isomorphisms which preserve M-ideals.

Example 2.1. Let $K$ be a square in the plane, and $S$ be the pentagon obtained from cutting off a corner of $K$. Let $T : A(K) \to A(S)$ be the restriction map, then $T$ and $T^{-1}$ preserve M-ideals since the only M-ideals in $A(K)$ and $A(S)$ are the trivial ones, as neither $K$ nor $S$ have any proper closed split face. We note that $T$ can be made to have arbitrary norm by cutting off a suitably sized corner of $K$.

Example 2.2. Let $K$ be a triangle in the plane, and $S$ be the quadrilateral obtained from cutting off the tip of $K$. Let $T : A(K) \to A(S)$ be the restriction map, then $T^{-1}$ preserves M-ideals since the only M-ideals in $A(S)$ are the trivial ones. However $T$ does not preserve M-ideals. For example, take $k \in \partial K$, then it’s annihilator $\{k\}^\perp$ in $A(K)$ is a proper M-ideal but $T(\{k\}^\perp)$ is not an M-ideal in $A(S)$, being neither the whole of $A(S)$ nor $\{0\}$.

The following example shows that an M-ideal preserving isomorphism between affine functions spaces on Bauer simplexes need not be an isometry.

Example 2.3. Let $X = [1, 2]$ and $Y = [3, 4]$. Let $T : C(X) \to C(Y)$ be defined by

$$Tf(y) = e^y f(y - 2) \quad (f \in C(X), \quad y \in Y).$$

Then $T$ is a linear isomorphism but not an isometry. In fact

$$\|Tf\| = \sup \{|e^y f(y - 2)| : y \in Y\}$$

$$= \sup \{|e^{x+2} f(x)| : x \in X\}$$

$$\leq e^4 \|f\|$$

and so $\|T\| \leq e^4$. Also $T^{-1} g(x) = e^{-(x+2)} g(x + 2)$ for all $x \in X$ and $\|T^{-1}\| \leq e^{-3}$. Now $T$ preserves M-ideals since if $J$ is an M-ideal in $C(X)$ with $J = F^\perp$ where $F$ is a closed subset of $X$, then $T(J) = G^\perp$ where $G = F + 2$ is a closed subset of $Y$. Likewise, $T^{-1}$ preserves M-ideals.
Let \( \mathcal{A} \) be a function algebra on a compact Hausdorff space \( X \), and \( S_\mathcal{A} \) be the state space of \( \mathcal{A} \). Let \( Z_\mathcal{A} = co(S_\mathcal{A} \cup -iS_\mathcal{A}) \) be the complex state space of \( \mathcal{A} \). The map \( \theta : \mathcal{A} \rightarrow A(Z_\mathcal{A}) \) defined by

\[
\theta f(z) = \text{re } z(f) \quad (f \in \mathcal{A}, z \in Z_\mathcal{A})
\]

is a real linear isomorphism [4, page 146]. Recall that a function algebra \( \mathcal{A} \) is antisymmetric if the conditions \( f \in \mathcal{A} \) and \( f \) is real-valued imply that \( f \) is constant [10, page 172]. Note that every extreme point of \( Z_\mathcal{A} \) is a split face [9].

Our next example is of a non-isometric M-ideal preserving linear isomorphism \( T : A(Z_\mathcal{A}) \rightarrow A(Z_\mathcal{A}) \).

**Example 2.4.** Let \( X = \overline{\Delta} \times [0, 1] \) where \( \Delta \) is the open unit disk in \( \mathbb{C} \). Let \( \mathcal{A} \) be the set of functions \( f \in C_\mathbb{C}(X) \) such that \( z \rightarrow f(z,t) \) is analytic in \( \Delta \), for each \( t \in [0,1] \). Then \( \mathcal{A} \) is not antisymmetric. (See [10, page 177],) Choose an element \( f \in \mathcal{A} \) such that \( f \) is real-valued, non-constant, and invertible. Define \( T : A(Z_\mathcal{A}) \rightarrow A(Z_\mathcal{A}) \) by \( Tg = \theta(f^{-1}g) \). Then \( T \) is a linear isomorphism and \( T^{-1} = \theta f^{-1} \theta^{-1} \). Let \( J \) be an M-ideal in \( A(Z_\mathcal{A}) \) with \( J = F^\perp \) where \( F \) is a closed split face of \( Z_\mathcal{A} \). For \( g \in J \) with \( g = \theta h \) and \( h \in \mathcal{A} \), we have

\[
Tg(z) = \text{re} f(z)z(h) = f(z)\text{re} h = f(z)g(z) = 0 \quad \text{for } z \in F \quad \text{and} \quad Tg \in F^\perp = J \quad \text{and} \quad TJ \subseteq J.
\]

Also \( J \supseteq T^{-1}(J) \). Indeed, for \( g \in J \) with \( g = \theta h \) and \( h \in \mathcal{A} \), we have

\[
T^{-1}g(z) = \theta(f^{-1}\theta^{-1}g(z)) = \theta(f^{-1}(z)h(z)) = \text{re}(f^{-1}(z)h(z)) = f^{-1}(z)\text{re} h(z) = f^{-1}(z)g(z) = 0 \quad \text{for} \quad z \in F.
\]

Thus \( T^{-1}g \in F^\perp = J \). So \( T \) preserves M-ideals. We note that \( T \) is not an isometry if \( \|f\| < 1 \), say.

### 3. Property M and a Banach-Stone Type Theorem for \( A(K) \)

It is known that an isometry between \( A(K) \) and \( A(S) \) always induces a homeomorphism between \( \partial K \) and \( \partial S \), for example see [8]. With this in mind, we ask what happens if we replace an isometry by a linear isomorphism? Chu and Cohen [7] have proved that if \( \partial K \) and \( \partial S \) are closed and every every extreme point of \( K \) and \( S \) is a split face, then a bound-2 isomorphism from \( A(K) \) to \( A(S) \) yields \( \partial K \) and \( \partial S \) homeomorphic. Jarosz [11] has proved an analogous result for function algebras, namely whenever there is a bound-2 complex linear isomorphism from \( \mathcal{A} \), then their Choquet boundaries, \( ch(\mathcal{A}) \) and \( ch(\mathcal{B}) \), are homeomorphic. For a function algebra \( \mathcal{A} \), \( \text{re} \mathcal{A} \) is linearly isometric to \( A(K) \), where \( \text{re} \mathcal{A} \) denotes the real part of \( \mathcal{A} \) and \( K \) is the state space of \( \mathcal{A} \), and in this setting, \( ch(\mathcal{A}) \) is homeomorphic to \( \partial K \) and every extreme point is a split face.

**Definition 3.1.** We say a linear isomorphism \( T \) from \( A(K) \) onto \( A(S) \) satisfies Property M if both \( T \) and \( T^{-1} \) preserve M-ideals.
We note that such a linear isomorphism need not be an isometry and in fact, it may have arbitrary bound, as examples above have shown. The class of surjective linear isomorphisms on a function space satisfying Property $M$ is thus larger than the class of isometries on it.

Our result in this section proves that if $T$ is a linear isomorphism from $A(K)$ onto $A(S)$ which satisfies Property $M$, and if every extreme point of $K$ and $S$ are split faces, then $\partial K$ is facially homeomorphic to $\partial S$. A key to the result lies in the well-known fact that an $M$-ideal in $A(K)$ is the annihilator of a closed split face of $K$.

Our first two lemmas give simple identifications of the maximal $M$-ideals in $A(K)$.

**Lemma 3.1.** Let $k \in \partial K$. If $\{k\}$ is a split face of $K$, then

$$J_k = \{f \in A(K) : f(k) = 0\}$$

is a maximal $M$-ideal.

**Proof.** Let $J$ be a proper $M$-ideal in $A(K)$ and $J \supseteq J_k$. Then $J^\perp$ is a split face of $K$ and

$$J^\perp \subseteq J_k^\perp = \{k\}^\perp = \{k\}$$

and so $J^\perp = \{k\}$. Hence $J = J_k^\perp = \{k\}^\perp$. So $J_k$ is maximal. \qed

**Lemma 3.2.** Suppose that every extreme point of $K$ is a split face. Then every maximal $M$-ideal in $A(K)$ is of the form $J_k$.

**Proof.** By Lemma 1.2, every $M$-ideal in $A(K)$ is of the form $J_F$, where $F$ is a closed split face of $K$ and

$$J_F = \{f \in A(K) : f|_F \equiv 0\}.$$ 

Suppose $J_F$ is a maximal $M$-ideal in $A(K)$. Let $k \in \partial F$. Then $\{k\}^\perp \supseteq J_F$ and by the maximality of $J_F$, we have $\{k\}^\perp = J_F$. \qed

Let $\text{Max}(A(K))$ be the set of all maximal $M$-ideals in $A(K)$. We topologise $\text{Max}(A(K))$ with the hull-kernel topology as follows.

Let $J \subseteq \text{Max}(A(K))$. We define the hull $\text{hull}(J)$ of $J$ to be:

$$\text{hull}(J) = \{M \in \text{Max}(A(K)) : M \supseteq J\}.$$ 

If $S \subseteq \text{Max}(A(K))$, then the kernel $\text{ker}(S)$ of $S$ is defined to be the largest $M$-ideal contained in $\cap\{J : J \in S\}$.

It can be shown that for $S \in \text{Max}(A(K))$, $\text{hull}(\text{ker}(S))$ defines the closure operation of a topology on $\text{Max}(A(K))$, called the hull-kernel topology (c.f. [4, page 225]).
We begin with the following lemma.

**Lemma 3.3.** Let $K$ and $S$ be compact convex sets and suppose every extreme point of $K$ and $S$ is a split face. Let $T : A(K) \to A(S)$ be a surjective linear isomorphism which satisfies Property M. Then the map $\Phi : \text{Max}(A(K)) \to \text{Max}(A(S))$ defined by

$$\Phi(J) = T(J), \text{ for } J \in \text{Max}(A(K)),$$

is a homeomorphism.

**Proof.** Since $T$ and $T^{-1}$ preserve M-ideals, then $T$ and $T^{-1}$ also preserve maximal M-ideals. Also $T$ is a bijection. Hence $\Phi$ is well-defined.

If $S \subseteq \text{Max}(A(K))$ then $T(\text{ker}(S))$ is an M-ideal and therefore,

$$T(\text{ker}(S)) \subseteq T(\cap\{J : J \in S\})$$

$$= \cap\{TJ : J \in S\}$$

$$= \cap\{\Phi J : J \in S\}$$

$$= \cap\{J : T^{-1}J \in S\},$$

which implies that $T(\text{ker}S) \subseteq \text{ker}(\Phi S)$. Applying the same argument to $T^{-1}$ and $\Phi^{-1}$ we have

$$T^{-1}(\text{ker}(\Phi(S))) \subseteq \text{ker}(\Phi^{-1}(\Phi(S))) = \text{ker}(S),$$

and so

$$\text{ker}(\Phi(S)) \subseteq T(\text{ker}(S))$$

and so we have equality. Thus, if $\overline{S}$ is the closure of $S$ in the hull-kernel topology, then we have

$$\Phi(\overline{S}) = \Phi(\{J : J \supseteq \text{ker}(S)\})$$

$$= \{TJ : J \supseteq \text{ker}(S)\}$$

$$= \{TJ : T(J) \supseteq \text{ker}(\Phi(S))\}$$

$$= \text{hull}(\text{ker}(\Phi(S))) = \overline{\Phi(S)}.$$

Thus $\Phi$ is a closed map. A similar argument shows that $\Phi^{-1}$ is closed, and so $\Phi$ is a homeomorphism. $\square$
Theorem 3.4. Let $K$ and $S$ be compact convex sets such that each extreme point in $K$ and $S$ are split faces. If there exists linear isomorphism $T : A(K) \to A(S)$ which satisfies Property $M$, then $\partial K$ is homeomorphic to $\partial S$ in the facial topology.

Proof. Let $\tau : \partial K \to \text{Max}(A(K))$ be defined by $\tau(k) = J_k$ for each $k \in \partial K$. Then $\tau$ is a bijection by Lemma 3.2. Let $S$ be a closed subset of $\text{Max}(A(K))$. Then $S = \text{hull}(\ker(S))$.

Let $F$ be the smallest closed split face of $K$ containing $\{k \in \partial K : \{k\}^\perp \in S\} = \{k \in \partial K : k \in \tau^{-1}(S)\}$. Then $F^\perp$ is the largest $M$-ideal contained in $\bigcap\{\{k\}^\perp : \{k\}^\perp \in S\}$; that is, $\ker(S) = F^\perp$. We show that $\tau^{-1}(S) = \partial F$ which is therefore facially closed. By definition of $F$, we have $\tau^{-1}(S) \subseteq \partial F$. Conversely, if $k \in \partial F$ then $\{k\} \subseteq F$ and so $\tau(k) = \{k\}^\perp \supseteq F^\perp$ hence $\tau(k) \in \text{hull}(\ker(S)) = S$. Thus $\tau^{-1}(S) \supseteq \partial F$. This proves that $\tau$ is continuous.

To show that $\tau$ is an open map, let $U$ be a facially open set in $\partial K$, then $U = \partial K \setminus \partial F$ where $F$ is a closed split face of $K$. We will show that $\text{Max}(A(K)) \setminus \tau(U)$ is closed, namely that $\text{Max}(A(K)) \setminus \tau(U)$ contains its hull-kernel. Now $\tau(U) = \{\{k\}^\perp : k \in U\}$ and so if $\{k\}^\perp \in \text{Max}(A(K)) \setminus \tau(U)$ then $k \notin U = \partial K \setminus \partial F$ and so $k \in \partial F$. That is,

$$\{ k : \{k\}^\perp \in \text{Max}(A(K)) \setminus \tau(U) \} \subseteq F.$$  

By definition of $\ker(\text{Max}(A(K)) \setminus \tau(U))$, its annihilator is the smallest closed split face containing $\{k : \{k\}^\perp \in \text{Max}(A(K)) \setminus \tau(U)\}$, and so

$$(\ker(\text{Max}(A(K)) \setminus \tau(U)))^\perp \subseteq F.$$  

Next, suppose

$$\{k\}^\perp \in \text{hull}(\ker(\text{Max}(A(K)) \setminus \tau(U))), \text{ for some } k \in \partial K.$$  

Then $\{k\}^\perp$ contains $\ker(\text{Max}(A(K)) \setminus \tau(U))$, and so,

$$\{k\} \subseteq (\ker(\text{Max}(A(K)) \setminus \tau(U)))^\perp \subseteq F.$$  

Hence $k \notin U$ and $\{k\}^\perp \in \text{Max}(A(K)) \setminus \tau(U)$. Therefore $\text{Max}(A(K)) \setminus \tau(U)$ is closed and $\tau$ is a homeomorphism.

Finally $\rho : \text{Max}(A(S)) \to \partial S$ defined by $\rho(J_s) = s$ for every maximal $M$-ideal in $\text{Max}(A(S))$ is a homeomorphism, as before.
Thus \( \sigma = \rho \circ \Phi \circ \tau \) is a homeomorphism from \( \partial K \) onto \( \partial S \), where \( \partial K \) and \( \partial S \) have the facial topology. \( \square \)

**Corollary 3.5.** Under the conditions of the above Theorem, the centre \( Z(A(K)) \) of \( A(K) \) is linearly isometric to \( Z(A(S)) \).

**Proof.** The homeomorphism \( \partial K \rightarrow \partial S \) induces a linear isometry between \( Z(A(K)) = C(\partial K) \) and \( Z(A(S)) = C(\partial S) \).

\( \square \)

Behrends [5, Ch7] gives three proofs of the classical Banach-Stone theorem using isometric invariants, one of which uses M-ideals. (It was Behrend’s book which was the starting point for this paper.) We can now give a re-formulation of this classical theorem, using the results in this paper.

**Corollary 3.6.** Let \( X \) and \( Y \) be compact Hausdorff spaces. If there exists a linear isomorphism \( T \) from \( C(X) \) onto \( C(Y) \) such that \( T \) satisfies Property \( M \), then \( X \) and \( Y \) are homeomorphic.

**Proof.** This follows from the fact that \( C(X) \) identifies with \( A(K) \), where \( K \) is a Bauer simplex and \( X \) is homeomorphic to \( \partial K \).

\( \square \)

**Remark 3.1.** The linear isomorphism \( T \) in the above theorem need not be an isometry (c.f. Example 2.3).

A natural question to is to ask whether there is any relationship between a homeomorphism in the facial topologies on \( \partial K \) and \( \partial S \) and a homeomorphism in the relative topologies. The answer is negative as the following simple examples show.

If \( \partial K \) and \( \partial S \) are relatively homeomorphic, it does not necessarily follow that they are facially homeomorphic.

**Example** 3.1. Let \( K \) be the unit square in \( \mathbb{R}^2 \) and \( S \) be the tetrahedron in \( \mathbb{R}^3 \). Then \( \partial K \) and \( \partial S \) are relatively homeomorphic. However, the facial topology on \( \partial K \) has the indiscrete topology whilst the facial topology on \( S \) is not indiscrete.

If \( \partial K \) and \( \partial S \) are facially homeomorphic, it does not necessarily follow that they are relatively homeomorphic.
Example 3.2. Let $K$ be a semi-circle in the plane, and $S$ be the bi-cone in $\mathbb{R}^3$ given by $S = \text{co} \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(1, 0, 1), (1, 0, -1)\}$. It is clear that there is a bijection between $\partial K$ and $\partial S$ and as their facial topologies are indiscrete, they are facially homeomorphic. However they are not relatively homeomorphic as $\partial K$ is closed but $\partial S$ is not.

4. Property $M$ : an M-ideal preserving map

This section contains numerous open questions and conjectures. Recall :

Definition 4.1. We say that a surjective linear isomorphism $T$ between Function spaces $E$ and $F$ satisfies Property $M$ if $T$ is M-ideal preserving, that is, both $T$ and $T^{-1}$ preserve M-ideals in $E$ and $F$.

As we have seen above in section 3, such a linear isomorphism need not be an isometry and may have arbitrary bound. The class of surjective linear isomorphisms on a Function space satisfying Property $M$ is thus strictly larger than the class of isometries on it.

The question of interest is: can we characterise Property $M$ in a natural way?

We consider two possible natural candidates, the Bounded Extension Property and the Best Approximant Property.

4.1. Bounded Extension Property. Let $X$ be compact $T_2$ space and $Y \subseteq X$ be closed. The pair $(H, L)$ of subspaces of $C(X)$ and $C(Y)$ respectively, has the bounded extension property (B.E.P.) if there is a constant $C$ such that for every $\varepsilon > 0$ and every open set $O \supseteq Y$ and every $f \in H$ there is a $g \in L$ such that

$$\|g\| \leq C\|f\|;$$

$$g|_Y = f;$$

$$|g(x)| \leq \varepsilon \|f\| \quad \forall x \in X\setminus O.$$ 

See for example, [12, Ch III.D]. The following lemma follows.

Lemma 4.1. If $(H, L)$ has the B.E.P. then

$$H_0 = \{f \in H : f|_X = 0\}$$

is an M-ideal in $H$.

The following seems natural.

Proposition 4.2. If $T : C(X) \rightarrow C(Y)$ is a linear isomorphism and $(H, L)$ has the B.E.P. then $T$ and $T^{-1}$ preserves M-ideals in $H$ and $L$. 
An analogous property in $A(K)$ would be as follows. Noting that, as there is a well-known B.E.P. for $A(K)$ (see, for example, [1, II.5]), we shall call this notion, the $M$-Bounded Extension Property (M.B.E.P.).

Let $K$ be compact convex set and $F \subset K$ be a closed split face. The pair $(H,L)$ of subspaces of $A(K)$ and $A(F)$ respectively, has the $M$-bounded extension property (M.B.E.P.) if there is a constant $C$ such that for every $\varepsilon > 0$ and every open set $O \supseteq F$ and every $f \in H$ there is a $g \in L$ such that

$$
\|g\| \leq C\|f\|; \\
g|_F = f; \\
|g(k)| \leq \varepsilon\|f\| \quad \forall k \in K \setminus O.
$$

The following lemma follows.

**Lemma 4.3.** If $(H,L)$ has the M.B.E.P. then

$$H_0 = \{f \in H : f|_F = 0\}$$

is an $M$-ideal in $H$.

**Question 1:** If $F$ has an extreme point $k$, then does the M.B.E.P. imply that $k$ is a weak peak point for $A(K)$? We conjecture affirmatively. (If $k$ is a w.p.p., this is a weaker condition than $k$ is a split face.)

[Recall $k \in \partial K$ is called a weak peak point for $A(K)$ if whenever $1 > \varepsilon > 0$ and $U$ is an open subset of $K$ then there is a function $h \in A(K)$ with $\|h\| \leq 1$, $h(k) > 1 - \varepsilon$ and $|h(x)| < \varepsilon$ for all $x \in \partial K \setminus U$.]

**Question 2:** Can we link this to a property that says a mapping $T$ is $M$-ideal preserving? We conjecture that this is likely.

4.2. **Best Approximant.** Let $E$ be a Banach space and let $J$ be an M-ideal in $E$. For each $\varphi \in E^*$ if there is one and only one $\varphi_0 \in J^\perp$ such that

$$
\|\varphi - \varphi_0\| = \inf\|\varphi - \varphi^\perp\|
$$

where the infimum is taken over all $\varphi^\perp \in J^\perp$, then we say $\varphi_0$ is the best approximant to $\varphi$.

**Lemma 4.4.** Let $J$ be an M-ideal in $A(K)$ and so $J^\perp$ is an $L$-summand in $A(K)^*$ with $P$ the associated $L$-projection on $A(K)^*$. Then each $\varphi \in A(K)^*$ has one and only one best approximant, namely $\varphi_0 = P(\varphi)$. 

**Question 3:** can we link this to a property that says a mapping $T$ is M-ideal preserving? We conjecture affirmatively. One idea would be as follows. Suppose $J \subset A(K)$ and $J' \subset A(S)$ are M-ideals and $T$ is a linear isomorphism between $A(K)$ and $A(S)$. Then $T^*$ will map each element $\varphi \in A(S)^*$ onto a unique point $\varphi_0 \in J'\perp$ which is of minimal distance from the given $\varphi$.

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